Fractional Jump-diffusion Pricing Model under Stochastic Interest Rate

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Abstract. The fractional jump-diffusion financial market model under stochastic interest rate is built, where stock price satisfies the fractional jump-diffusion process, and interest rate satisfies the fractional Hull-White model. By mean of the physical probabilistic measure of price process, the principle of fair premium, and fractional jump-diffusion process theory, the pricing formulae of European option are obtained. The European option pricing model is generalized.

Keywords: option pricing; actuarial method; fractional jump-diffusion process; stochastic interest rate.

1. Introduction

Recently, some scholar have focused on the valuation of European options when the underlying value follows a jump-diffusion process or Levy processes which are a fairly large class of continuous time processes with stationary independent increments. Chen (2008) studied the valuation of European options when the underlying value follows a jump diffusion process. Fajardo and Mordecki (2006) discussed derivative pricing on two risky asset prices driven by two-dimensional Levy processes. While Levy process allows for fat tails and jumps, it does not allow for long-range dependence property. Fractional Brownian motion has been considered as a suitable tool in financial models because it not only allows for fat tails, but also allows for long-range dependence property, see Elliot and Hoek (2003), Hu and Øksendal (2003), Bjork and Hult (2005), Guasoni (2006).

Bladt and Rydberg (1998) introduced a new method of option pricing, using physical probability measure of price process and the principle of fair premium, they deal with the problems of option pricing under the unbalance, arbitrage existing and incomplete circumstance, and transform option pricing into a problem of equivalent and fair insurance premium. Xue and Sun (2009) discussed European option under fractional jump-diffusion Ornstein-Uhlenbeck model, Xue and Li (2010) studied options pricing on the maximum or minimum of two risky assets in fractional Brownian motion environment by actuarial approach.

In this paper, we consider the European option pricing problem in fractional jump-diffusion process and stochastic interest rate. In section 2, assume that the stock price obeys the stochastic differential equation driven by fractional Brownian motion and compound Poisson process, we present the fractional jump-diffusion financial market model with stochastic interest rate. In section 3, using physical probability measure and the principle of fair premium, we obtain the pricing formula of the European option.

2. Fractional Jump-diffusion Market Model with Stochastic Interest Rate

Consider the following model (A): the dynamics of the stock price process and interest rate do satisfy the following stochastic differential equations

\[ dr_t = (b - ar_t)dt + cdB^H_t, \]
\[ dS_t = S_t \{ (\mu - \lambda \theta)dt + \sigma d\hat{W}^H_t + dJ_t \}, \]

where \( B^H_t \) and \( \hat{W}^H_t \) are fractional Brownian motions with Hurst index \( H \) and \( \hat{H} \) respectively, and \( J_t \) is a compound Poisson process.

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where $a, b, c, \mu, \sigma$ are constant, \{\hat{B}^H_t, t \geq 0\}, \{\hat{W}^H_t, t \geq 0\}$ are the fractional Brownian motion on the complete probability space $(\Omega, \mathcal{F}, P)$ with correlation coefficient $\delta$, and \{\hat{J}_t, t \geq 0\} is a jump process whose jump times are specified by the jump times of the Poisson process \{N_t, t \geq 0\} with intensity $\lambda$, and the size of the $i^{th}$ jump is $U_i$. In other words, the process \{\hat{J}_t, t \geq 0\} is a compounded Poisson process

\[
\{\hat{J}_t, t \geq 0\} = \sum_{i=1}^{N(t)} U_i
\]

\[
\text{where } U_i \text{ is a sequence of independent identically distributed random variables with the finite expected value } \theta = \mathbb{E}(U_i), \ U_i > 0, (i = 1, 2, \cdots)
\]

Assume that \{\hat{B}^H_t, t \geq 0\} and \{\hat{W}^H_t, t \geq 0\} are independent to \{N_t, t \geq 0\} and \{\hat{J}_t, t \geq 0\].

Suppose that \{\hat{B}^H_t, \hat{W}^H_t\} is the two dimension fractional Brownian motion on $(\Omega, \mathcal{F}, P)$. Let

\[
\hat{B}_t^H = B_t^H, \quad \hat{W}_t^H = \delta B_t^H + \sqrt{1-\delta^2} W_t^H,
\]

then \{\hat{B}^H_t, t \geq 0\}, \{\hat{W}^H_t, t \geq 0\} are the fractional Brownian motion on $(\Omega, \mathcal{F}, P)$ with correlation coefficient $\delta$. We get the model (B):

\[
dr_t = (b - ar_t) dt + c dB^H_t, \quad \text{ (3)}
\]

\[
dS_t = S_t \{[\mu - \lambda \theta] dt + \sigma \delta dB^H_t + \sigma \sqrt{1 - \delta^2} dW^H_t + dJ_t\}. \quad \text{ (4)}
\]

**Theorem 2.1** The solution of stochastic differential equation (3) is

\[
r_t = r_0 e^{-\alpha(t-t_0)} + \frac{b}{a} + c \int_{t_0}^{t} e^{\alpha(t-t')} dB^H_t. \quad \text{ (5)}
\]

**Proof:** By the fractional Ito formula, we have

\[
d(\exp\{a(t-t_0)\} r_t) = a \exp\{a(t-t_0)\} r_t dt + \exp\{a(t-t_0)\} dr_t
\]

\[
= \exp\{a(t-t_0)\} (b dt + c dB^H_t).
\]

Then we get the result.

**Theorem 2.2**[10] The solution for stochastic differential equation (4) is

\[
S_t = S_{t_0} \prod_{t_{i-1}}^{t} \left(1 + U_i\right) \exp\{[\mu - \lambda \theta] (t - t_0) - \frac{\sigma^2}{2} (t - t_0)^2 - \frac{\sigma \delta}{2} (B^H_t - B^H_{t_0}) + \sigma \sqrt{1 - \delta^2} (W^H_t - W^H_{t_0})\}. \quad \text{ (6)}
\]

We consider the European call option with payoff $(S_t - K)^+$, where $T$ is maturity date, $K$ is exercise price.

**Definition 2.3**[9] The expectation return rate $\beta_\nu$ of $\{S_t, t \geq 0\}$ on $[t, T]$ is defined by

\[
\exp\left\{\int_{t}^{T} \beta_\nu du\right\} = \frac{\mathbb{E}(S(T))}{S(t)}. \quad \text{ (7)}
\]

**Theorem 2.4**[10] The expectation return rate $\beta_\nu$ of the stock price $\{S_t, t \geq 0\}$ on $[t, T]$ satisfies $\beta_\nu = \mu$.

3. **European Option Pricing Formula**

**Definition 3.1**[10] The actuarial price of European option $(S_T - K)^+$ be defined by

\[
c(t, S_t, K, T) = E\{\exp\left\{-\int_{t}^{T} \beta_\nu du\right\} S_T - \exp\left\{-\int_{t}^{T} r_{u} du\right\} K\} I_{\{\exp\left\{-\int_{t}^{T} r_{u} du\right\} > \exp\left\{-\int_{t}^{T} \beta_\nu du\right\} K\}}.
\]

**Lemma 3.2**[4] Let $a, b, c, d, k$ be real numbers, and assume that $\xi_1, \xi_2$ are standard normal random variable with $\text{cov}(\xi_1, \xi_2) = \rho$, then

\[
E\{\exp\{c\xi_1 + d\xi_2\} I_{\{a\xi_1 + b\xi_2 > k\}}\} = \exp\left\{\frac{1}{2} \left(c^2 + d^2 + 2 \rho cd\right) \Phi\left(\frac{ac + bd + \rho(ad + bc) - k}{\sqrt{a^2 + b^2 + 2 \rho ab}}\right)\right\},
\]

where $\Phi(\cdot)$ is standard normal distribution function.

**Lemma 3.3** Let $a, b, c, k$ be real numbers, and assume that $\xi_1, \xi_2, \xi_3$ are standard normal random variable and whose co-variance is given by:
\[
\text{cov}(\xi_1, \xi_3) = \rho, \text{cov}(\xi_1, \xi_2) = 0, \text{cov}(\xi_2, \xi_3) = 0,
\]
then
\[
E\{\exp\{a\xi_1 + b\xi_2\}I_{\{a\xi_1 + b\xi_2 + c\xi_2 > 2k\}}\} = \exp\left\{\frac{a^2 + b^2}{2}\right\} \Phi\left(\frac{a^2 + b^2 + \rho ac - k}{\sqrt{a^2 + b^2 + c^2 + 2 \rho ac}}\right).
\]

**Proof:** Let \(w_1 = \frac{a\xi_1 + b\xi_2}{\sqrt{a^2 + b^2}}, w_2 = \xi_3\), then
\[
E[w_1^2] = 1, E[w_2] = 1, E[w_1w_2] = \frac{\rho b}{\sqrt{a^2 + b^2}}.
\]

By lemma 3.2, we get
\[
E\{\exp\{a\xi_1 + b\xi_2\}I_{\{a\xi_1 + b\xi_2 + c\xi_2 > 2k\}}\} = E\{\exp\left\{\sqrt{a^2 + b^2} w_1\right\}I_{\{\sqrt{a^2 + b^2} w_1 + c w_2 > 2k\}}\}
= \exp\left\{\frac{a^2 + b^2}{2}\right\} \Phi\left(\frac{a^2 + b^2 + c^2 + 2 \rho bc - k}{\sqrt{a^2 + b^2 + c^2 + 2 \rho bc}}\right).
\]

**Theorem 3.4** The actuarial price of European call option with payoff \((S_T - K)^+\) is
\[
c = S_t \exp\{-\lambda \theta (T - t)\} \sum_{n=0}^{\infty} \left\{\frac{[\lambda(T-t)]^n}{n!} E\left[\prod_{j=0}^{n} (1 + U_j) \Phi(d_1^{(n)})\right]\right\} - K \exp\left\{-\frac{r}{a}(1 - e^{-a(T-t)}) - \frac{b}{a}(T-t) + \frac{D_1}{2}\right\} \sum_{n=0}^{\infty} \left\{\frac{[\lambda(T-t)]^n}{n!} E\left[\Phi(d_2^{(n)})\right]\right\},
\]
where \(\Phi(x)\) is standard normal distribution function, and
d_1^{(n)} = \frac{\ln S_t + \frac{r}{a}(1 - e^{-a(T-t)}) + \frac{b}{a}(T-t) - \lambda \theta (T-t) + \frac{D_1 + D_2}{2}}{\sqrt{D_1 + D_2 + D_3 + 2 D_4}},
d_2^{(n)} = \frac{\ln S_t + \frac{r}{a}(1 - e^{-a(T-t)}) + \frac{b}{a}(T-t) - \lambda \theta (T-t) - \frac{D_1 + D_2}{2}}{\sqrt{D_1 + D_2 + D_3 + 2 D_4}},
D_1 = (1 - \delta^2) \sigma^2 (T^{2H} - t^{2H}),
D_2 = \delta^2 \sigma^2 (T^{2H} - t^{2H}),
D_3 = c^2 \int_{T}^{T} [M_H((T-\mu)e^{(\mu-\lambda)})]^2 \, d\mu,
D_4 = \delta \sigma \int_{T}^{T} \int_{T}^{T} M_H(I_{[T,T]} M_H((T-\mu)e^{(\mu-\lambda)}) d\mu d\tau,
and operator \(M_H\) is in [5].

**Proof:** Let
d_n = \ln S_t + \frac{r}{a}(1 - e^{-a(T-t)}) + \frac{b}{a}(T-t) - \lambda \theta (T-t) - \frac{\sigma^2}{2} (T^{2H} - t^{2H}) + \ln \prod_{j=0}^{n} (1 + U_j),
A = \{\exp\{-\int_{T}^{T} \beta(u) \, du\} S_T > \exp\{-\int_{T}^{T} r_u \, du\} K\}.

By theorem 2.1, 2.2 and 2.4, we have
\[
\exp\{-\int_{T}^{T} \beta_u \, du\} S_T = S_T \prod_{j=0}^{n} (1 + U_j) \exp\{-\lambda \theta (T-t) - \frac{\sigma^2}{2} (T^{2H} - t^{2H}) + \delta \sigma (b_H^u - b_H^u) + \sigma \sqrt{1 - \delta^2} (W_H^u - W_H^u)\},
\]
\[
\exp\{-\int_{T}^{T} r_u \, du\} K = K \exp\{-\frac{r}{a}(1 - e^{-a(T-t)}) - \frac{b}{a}(T-t) - \int_{T}^{T} e^{(\mu-\lambda)} \, dB_H^u \, d\tau\}.
\]
By fractional Ito formula, we get
\[ \int_{T}^{T} e^{a(t-u)} dB^{H}_{u} \, d\tau = \int_{T}^{T} (T-u) e^{a(t-u)} dB^{H}_{u} . \]

Let
\[ \xi = \sigma \sqrt{1 - \delta^2} (W^{H}_{T} - W^{H}_{t}) + \sigma \delta (B^{H}_{T} - B^{H}_{t}) + c \int_{t}^{T} (T-u) e^{a(u-t)} dB^{H}_{u} , \]

then \( \exp\{- \int_{t}^{T} \beta_{u} du\} S_{t} > \exp\{- \int_{t}^{T} r_{u} du\} K \) is equal to
\[ \xi = \ln K + \frac{\sigma}{a} (1 - e^{-a(T-t)}) - \frac{b}{a} (T-t) + \nu \theta (T-t) + \frac{\sigma^2}{2} (T^{2H} - t^{2H}) - \ln \prod_{n} (1+U_{i}). \]

Hence
\[ \begin{align*}
 c &= E\{ \exp\{ \int_{t}^{T} \beta_{u} du\} S_{T} - \exp\{ \int_{t}^{T} r_{u} du\} K \} \\
 &= E\{ E\{ \exp\{ \int_{t}^{T} \beta_{u} du\} S_{T} - \exp\{ \int_{t}^{T} r_{u} du\} K \mid N_{T-} \} \} \\
 &= \sum_{n=1}^{\infty} \left[ \Lambda(T-t) \right]^{n} e^{-\Lambda(T-t)} n! E\{ \exp\{ \int_{t}^{T} \beta_{u} du\} S_{T} - \exp\{ \int_{t}^{T} r_{u} du\} K \mid N_{T-} = n \} \\
 &= \sum_{n=1}^{\infty} \left[ \Lambda(T-t) \right]^{n} e^{-\Lambda(T-t)} n! [\Pi_{1} - \Pi_{2}] ,
\end{align*} \]

where
\[ \begin{align*}
 \Pi_{1} &= E\{ \exp\{ \int_{t}^{T} \beta_{u} du\} S_{T} I_{[\xi_{T-} < \xi]} \mid N_{T-} = n \} \\
 &= EE\{ \exp\{ \int_{t}^{T} \beta_{u} du\} S_{T} \mid N_{T-} = n, U_{i}, i = 1, 2, \ldots \} \\
 &= E(S \prod_{i=1}^{n} (1+U_{i}) \exp\{ -\lambda \theta (T-t) - \frac{\sigma^2}{2} (T^{2H} - t^{2H}) \} E\{ \exp|\sigma \delta (B^{H}_{T} - B^{H}_{t}) + c \int_{t}^{T} (T-u) e^{a(u-t)} dB^{H}_{u} | N_{T-} = n, U_{i}, i = 1, 2, \ldots \} \\
 &= S \exp\{ -\lambda \theta (T-t) \} E\prod_{i=0}^{n} (1+U_{i}) \Phi(\frac{d_{a} + D_{1} + D_{2} + D_{4}}{\sqrt{D_{1} + D_{2} + D_{3} + 2D_{4}}}) \\
 &= S \exp\{ -\lambda \theta (T-t) \} E\prod_{i=0}^{n} (1+U_{i}) \Phi(d_{a}^{(n)}) ,
\end{align*} \]

\[ \begin{align*}
 \Pi_{2} &= K \exp\{ \frac{r}{a} (1 - e^{-a(t-t)}) - \frac{b}{a} (T-t) \} E\{ \exp\{ -c \int_{T}^{T} (T-u) e^{a(u-t)} dB^{H}_{u} \} I_{[\xi_{T-} = n]} \mid N_{T-} = n \} \\
 &= K \exp\{ \frac{r}{a} (1 - e^{-a(t-t)}) - \frac{b}{a} (T-t) \} E\{ E\{ \exp\{ -c \int_{T}^{T} (T-u) e^{a(u-t)} dB^{H}_{u} \} I_{[\xi_{T-} = n]} \mid N_{T-} = n, U_{i}, i = 1, 2, \ldots \} \\
 &= K \exp\{ \frac{r}{a} (1 - e^{-a(t-t)}) - \frac{b}{a} (T-t) + D_{a} \} E\{ \Phi(\frac{d_{a} + D_{1} + D_{4}}{\sqrt{D_{1} + D_{2} + D_{3} + 2D_{4}}}) \} \\
 &= K \exp\{ \frac{r}{a} (1 - e^{-a(t-t)}) - \frac{b}{a} (T-t) + D_{a} \} E\{ \Phi(d_{a}^{(n)}) \} .
\end{align*} \]

**Remark 3.5** When \( b = 0 \), \( c = 0 \), \( a \to 0 \), we get the price of European call option in the fractional jump-diffusion environment
\[ c = S \exp\{ -\lambda \theta (T-t) \} \sum_{n=0}^{\infty} \left[ \Lambda(T-t) \right]^{n} e^{-\Lambda(T-t)} n! E\prod_{i=0}^{n} (1+U_{i}) \Phi(d_{a}^{(n)}) . \]
\[-K \exp\{-r(T-t)\} \sum_{i=0}^{n} \left[ \lambda(T-t)^i e^{\lambda(T-t)} \right] \frac{1}{n!} E[\Phi(d^{(n)}_2)], \]  

(9)

where \( \Phi(x) \) is standard normal distribution function, and

\[
d_1^{(n)} = \frac{\ln \left( \frac{S}{K} \right) + r(T-t) - \lambda \theta (T-t) + \frac{\sigma^2}{2} (T^{2H} - t^{2H}) + \ln \prod_{i=0}^{n} (1+U_i)}{\sigma \sqrt{T^{2H} - t^{2H}}},
\]

\[
d_2^{(n)} = \frac{\ln \left( \frac{S}{K} \right) + r(T-t) - \lambda \theta (T-t) - \frac{\sigma^2}{2} (T^{2H} - t^{2H}) + \ln \prod_{i=0}^{n} (1+U_i)}{\sigma \sqrt{T^{2H} - t^{2H}}}.\]

In particular, When \( \lambda = 0, U_i = 0, (i = 1, 2, \cdots) \), we get the price of European call option in the fractional Brownian motion environment

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5. References


